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Laplacian energy of trees with at most 10 vertices

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Abstract

Let T_n be the set of all trees with $n \leq 10$ vertices. We show that the Laplacian energy of any tree T_n is strictly between the Laplacian energy of the path P_n and the star S_n, the authors partially proving that the conjecture hold for any tree T_n, where $n \leq 10$.

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1. Introduction and preliminaries

Let $G = (V, E)$ be a finite, simple and undirected graph with vertices $V = \{1, 2, ..., n\}$ and $m = |E|$ edges. The degree of a vertex $u \in V$ will be denoted by d_n. Let G have adjacency matrix A with eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, and the Laplacian matrix $L = D - A$, where D is the diagonal matrix of vertex degrees, with eigenvalues $\mu_1 \geq \mu_2 \geq ... \geq \mu_n = 0$. Additional details on the theory of graph spectra may be found in [1].

The energy and Laplacian energy of G are defined as follows

$$E(G) = \sum_{i=1}^{n} |\lambda_i|, \quad LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|.$$

The energy of a graph was defined by Ivan Gutman in [2] and it has a long known chemical applications; for details see surveys [3, 4]. On the other hand, the Laplacian energy of a graph G,

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defined by Ivan Gutman and Zhou in [5].

An important problem in the area of spectral graph theory is to determine which graph G among all graphs with n vertices has the maximum energy $E(G)$. Among the trees (a tree T is a connected undirected graph without cycles), it has been long known [7] that for positive integer n, the path P_n has maximum energy and the star S_n has minimum energy. For the Laplacian energy, few results are established, so that the extremal energy graphs are not known even for trees. In [8], Radenković and Gutman studied the correlation between the energy and Laplacian energy of trees. In that paper, the energy and the Laplacian energy for trees are computed. They found that the energy and the Laplacian energy of a tree are inversely proportional, and formulated the following conjecture:

Conjecture 1. *Let T_n be a tree on n vertices. Then*
$LE(P_n) \leq LE(T_n) \leq LE(S_n)$.

Furthermore, in [9], Trevisan et al.,(2011) showed that the above conjecture is true for the restricted class of trees, namely those whose diameter is 3. Our goal here is to show that the Conjecture 1.1, is also true for all trees with at most 10 vertices. The plan of the paper is as follows: In the next section we will first give a notation to name trees. Next, we establish a result (Lemma 2.1) about trees with at most 10 vertices, Theorem 2.2, gives the fact that the Conjecture 1.1, is true for all trees with at most 10 vertices. We will conclude the paper with a discussion about conclusions and future work.

2. Main results

2.0.1. Names of general trees

In tables below, we need a notation to name trees. Given a tree, pick some vertex and call it the $root$. Now walk along the tree (depth-first), starting at the root, and when a vertex is encountered for the first time, write down its distance to the root. The sequence of integers obtained is called a $level\ sequence$ for the tree. A tree is uniquely determined by any level sequence. The parent of a vertex labeled m is the last vertex encountered earlier that was labeled $m-1$. For example, the tree $K_{1,9}$ gets level sequence 0111111111 if the vertex of degree 9 is chosen as root, 0122222222 otherwise. We use exponent to indicate repetition: 0111111111 can be written 01^9 and 0121212 as $0(12)^3$.

Figure 1. $0(12)^2 1(23)^2$

Lemma 2.1. *There are exactly 199 trees with at most 10 vertices.*

Proof. If there is a tree with no edge, then it is a single vertex (it is $\#0$ on the TABLE 1). Moreover, from the appendix (table# 2) of [1], one can find all trees T_n for $2 \leq n \leq 10$, where n is the number of vertices in T. □

Now we are ready to discuss the main result of the paper.

Theorem 2.1. *The conjecture* $LE(P_n) \leq LE(T_n) \leq LE(S_n)$ *is true for* $1 \leq n \leq 10$.

Proof. By Lemma 2.1, we have all trees with at most 10 vertices, we give all these trees in third column on the tables below in the form of level sequence. Now, from section 1, we have $L = D - A$ the Laplacian matrix of T, and $LE(T) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$ is the Laplacian energy of T. By direct calculation (one can do this exercise by computer, by use of suitable mathematical softwares, for example Matlab or Mathematica) we find the Laplacian spectrum of T (the set of eigenvalues of the matrix L, where the exponent of an eigenvalue denotes the multiplicity of the corresponding eigenvalue), and Laplacian energy of each T. Here, we note that on TABLE 1, TABLE 2, TABLE 3 and TABLE 4 we give all trees in such a way that the star S_n appear first and the path P_n appear last and all other trees appear between S_n and P_n for the same n.

TABLE 1, TABLE 2, TABLE 3 and TABLE 4 gives a serial number, the number of vertices n, a level sequence, the Laplacian spectrum and the Laplacian energy of T. For $n = 1, 2, 3$ we can see from TABLE 1, that $LE(P_n) = LE(T_n) = LE(S_n)$. For $4 \leq n \leq 10$ from TABLE 1, TABLE 2, TABLE 3 and TABLE 4 it is easy to observe that $LE(P_n) \leq LE(T_n) \leq LE(S_n)$, with equality if and only if $P_n = T_n = S_n$.

Table 1.

#	n	trees	Laplacian spectrum	$LE(T_n)$
0	0	0	0	0
1	2	01	0, 2	2
2	3	01^2	0, 1,3	3.3333
3	4	01^3	0, 1^2,4	5
4	4	0112	0, 0.5858, 2, 3.4142	4.8284
5	5	01^4	0, 1^3, 5	6.8000
6	5	01^212	0, 0.5188, 1, 2.3111, 4.1701	6.5624
7	5	$0(12)^2$	0, 0.3820, 1.3820, 2.6180, 3.6180	6.0720
8	6	01^5	0, 1^4, 6	8.6667
9	6	01^312	0, 0.4859, 1^2, 2.4280, 5.0861	8.3615
10	6	01^212^2	0, 0.4384, 1^2, 3, 4	8.9616
11	6	01^2123	0, 0.3249, 1, 1.4608, 3, 4.2143	7.7619
12	6	$01(12)^2$	0, 0.3820, 0.6972, 2, 2.6180, 4.3028	7.8416
13	6	012123	0, 0.2679, 1, 2, 3, 3.7321	7.4642

Table 1. (*cont.*)

#	n	trees	Laplacian spectrum	$LE(T_n)$
14	7	01^6	$0, 1^5, 7$	10.5714
15	7	01^412	$0, 0.4659, 1^3, 2.4827, 6.0514$	10.2111
16	7	01^312^2	$0, 0.3983, 1^3, 3.3399, 5.2618$	10.3463
17	7	01^3123	$0, 0.2955, 1^2, 1.4911, 3.1169, 5.0965$	9.5697
18	7	$01^2(12)^2$	0, 0.3820, 0.6086, 1, 2.2271, 2.6180, 5.1642	9.7330
19	7	01^2123^2	$0, 0.2679, 1^2, 1.5858, 3.7321, 4.4142$	9.4355
20	7	01^21223	0, 0.3217, 0.6802, 1, 2.1397, 3.2297, 4.6287	9.7105
21	7	01^21234	$0, 0.2254, 1^2, 2.1859, 3.3604, 4.2283$	9.2635
22	7	0112123	0, 0.2603, 0.6262, 1.4055, 2.2742, 3.0996, 4.3342	9.1303
23	7	$0(12)^3$	$0, 0.3820, 0.3820, 1.5858, 2.6180^2, 4.4142$	9.0147
24	7	$0(123)^2$	0, 0.1981, 0.7530, 1.5550, 2.4450, 3.2470, 3.8019	8.7021
25	8	01^7	$0, 1^6, 8$	12.5000
26	8	01^512	$0, 0.4525, 1^4, 2.5135, 7.0340$	12.0950
27	8	01^412^2	$0, 0.3738, 1^4, 3.4849, 6.1413$	12.2524
28	8	01^312^3	$0, 0.3542, 1^4, 4, 5.6458$	12.2616
29	8	01^4123	$0, 0.2774, 1^4, 3.1610, 6.0548$	11.9384
30	8	$01^3(12)^2$	$0, 0.3820, 0.5607, 1^2, 2.3389, 2.6180, 6.1004$	11.6146
31	8	01^3123^2	$0, 0.2384, 1^3, 1.6367, 4, 5.1249$	11.2488
32	8	01^31223	$0, 0.2888, 0.6742, 1^2, 2.1694, 3.5857, 5.2819$	11.8740
33	8	01^212^223	$0, 0.3187, 0.5858, 1^2, 2.3579, 3.4142, 5.3234$	11.6910
34	8	01^31234	$0, 0.023, 1^3, 2.2472, 3.4527, 5.0979$	11.0955
35	8	01^212123	0, 0.2538, 0.5472, 1, 1.4689, 2.4066, 3.1504, 5.1732	10.9603
36	8	$01(12)^3$	$0, 0.3820^2, 0.7639, 2, 2.6180, 2.6180, 5.2361$	10.9442
37	8	012^2123^2	$0, 0.1864, 1^3, 2.4707, 4, 4.3429$	11.1272
38	8	012^21223	0, 0.2137, 0.6177, 1, 1.4977, 2.3537, 3.8408, 4.4763	10.8417
39	8	01^212234	0, 0.2243, 0.5858, 1, 1.4108, 2.7237, 3.4142, 4.6412	11.0582
40	8	$01^21(23)^2$	0, 0.3065, 0.3820, 1, 1.6703, 2.6180, 3.3297, 4.6935	10.7824
41	8	01121223	0, 0.2509, 0.5858, 0.7287, 2, 2.3349, 3.4142, 4.6855	10.8692
42	8	01^212345	0, 0.1667, 0.7276, 1, 1.6353, 2.6729, 3.5643, 4.2332	10.4408
43	8	$01(123)^2$	0, 0.1981, 0.4915, 1.3204, 1.5550, 2.8258, 3.2470, 4.3623	10.3701
44	8	01121234	0, 0.1864, 0.5858, 1, 2, 2.4707, 3.4142, 4.3429	10.4556
45	8	$0(12)^2123$	0, 0.2434, 0.3820, 1.1798, 2, 2.6180, 3.1386, 4.4383	10.3897
46	8	01231234	0, 0.1522, 0.5858, 1.2346, 2, 2.7654, 3.4142, 3.8478	10.0548
47	9	01^8	$0, 1^7, 9$	14.4440
48	9	01^612	$0, 0.4428, 1^5, 2.5330, 8.0242$	14.0033
49	9	01^512^2	$0, 0.3572, 1^5, 3.5554, 7.0874$	14.1745

Table 2.

#	n	trees	Laplacian spectrum	$LE(T_n)$
50	9	01^5123	$0, 0.2650, 1^5, 3.1832, 7.0355$	13.8426
51	9	$01^4(12)^2$	$0, 0.3820, 0.5300, 1^3, 2.4027, 2.6180, 7.0672$	13.5092
52	9	01^412^3	$0, 0.3272\ 1^5, 4.3519, 6.3209$	14.2345
53	9	01^4123^2	$0, 0.2201, 1^4, 1.6634, 4.0549, 6.0615$	13.1218
54	9	01^41223	$0, 0.2679, 0.6711\ 1^3, 2.1814, 3.7321, 6.1474$	13.4552
55	9	01^31212^2	$0, 0.3158, 0.5356, 1^3, 2.4475, 3.5152, 6.1860$	13.6302
56	9	$0(12^3)^2$	$0, 0.2087, 1^4\ 1.6972, 4.7913, 5.3028$	13.0771
57	9	01^41234	$0, 0.1876, 1^4, 2.2755, 3.4819, 6.0550$	12.9581
58	9	01^312^223	$0, 0.2825, 1^3, 2.3735, 4.0864, 5.6793$	13.6116
59	9	01^312123	$0, 0.2483, 0.5063, 1^2, 1.4950, 2.4702, 3.1767, 6.1036$	12.8342
60	9	$01^2(12)^3$	$0, 0.3820^2, 0.6711, 1, 2.1814, 2.6180^2, 6.1474$	12.9075
61	9	0112^212^2	$0, 0.2377, 0.6484, 1^3, 2.6501, 4.3124, 5.3314$	13.5611
62	9	$01^2(12^2)^2$	$0, 0.2679, 0.5505, 1^3, 3, 3.7321, 5.4495$	13.6965
63	9	012^2123^3	$0, 0.1649, 1^4, 2.5680, 4.1652, 5.1019$	13.0035
64	9	0122312^3	$0, 0.1884, 0.6144, 1^2, 1.5333, 2.3798, 4.1545, 5.1296$	12.6611
65	9	012^212^223	$0, 0.2043, 0.5405, 1^2, 1.5989, 2.4425, 4.0170, 5.1969$	12.6460
66	9	0112312^3	$0, 0.2022, 0.5693, 1^2, 1.4124, 2.8273, 3.7046, 5.2842$	13.0304
67	9	$0(12)^212^3$	$0, 0.2679, 0.3820, 1^2, 1.6972, 2.6180, 3.7321, 5.3028$	12.6391
68	9	01^212^2123	$0, 0.2232, 0.4919, 1^2, 1.4712, 3, 3.4838, 5.3298$	12.9606
69	9	$01^2121223$	$0, 0.2427, 0.5371, 0.6893, 1, 2.1297, 2.4166, 3.6434, 5.3411$	12.8395
70	9	$01(12)^212^2$	$0, 0.3047, 0.3820, 0.7566, 1, 2.0960, 2.6180, 3.4609, 5.3818$	12.8912
71	9	01^31234	$0, 0.1487, 0.7169, 1^2, 1.6629, 2.7405, 3.6330, 5.0980$	12.2763
72	9	01^212334^2	$0, 0.1830, 0.5723, 1^2, 1.5095, 3, 4.0444, 4.6907$	12.8036
73	9	01^212323^2	$0, 0.2679, 0.3446, 1^2, 1.7892, 3, 3.7321, 4.8662$	12.5528
74	9	$01^2(123)^2$	$0, 0.1981, 0.4116, 1, 1.4064, 1.5550, 3, 3.2470, 5.1819$	12.1929
75	9	01122312^2	$0, 0.2118, 0.5546, 0.7223, 1, 2.0782, 2.7338, 3.8525, 4.8468$	12.8004
76	9	$01^2121234$	$0, 0.1774, 0.5242, 1^2, 2.1609, 2.4961, 3.4670, 5.1743$	12.3745
77	9	$01(12)^2123$	$0, 0.2398, 0.3820, 0.7199, 1.4240, 2.2032, 2.6180, 3.1692, 5.2439$	12.2464
78	9	$0(12)^4$	$0, 0.3820, 0.3820, 0.3820, 1.6972, 2.6180, 2.6180^2, 5.3028$	12.0914
79	9	$0(123^2)^2$	$0, 0.1392, 0.6972, 1^2, 1.7459, 3, 4.1149, 4.3028$	12.1887
80	9	01223412^2	$0, 0.1658, 0.4679, 1, 1.3434, 1.6527, 3, 3.8794, 4.4909$	12.0738
81	9	01223123^2	$0, 0.1538, 0.5764, 1^2, 2.1128, 2.6757, 4.0748, 4.4065$	12.3174
82	9	01123412^2	$0, 0.1627, 0.5321, 1^2, 2.0892, 3, 3.5723, 4.6437$	12.3882
83	9	$012^21(23)^2$	$0, 0.1953, 0.3820, 1, 1.2108, 2.1449, 2.6180, 3.9064, 4.5426$	12.2016
84	9	$0(1223)^2$	$0, 0.1729, 0.5587, 0.6617, 1.4331, 2.2091, 2.4851, 3.9563, 4.5231$	12.1250
85	9	012^212123	$0, 0.2217, 0.3327, 1, 1.1923, 2.1071, 3, 3.4413, 4.7049$	12.2844
86	9	0112312123	$0, 0.1862, 0.4822, 0.7043, 1.4073, 2.1338, 2.8532, 3.5372, 4.6958$	12.2178
87	9	$01121(23)^2$	$0, 0.2311, 0.3820, 0.6416, 1.6129, 2.2591, 2.6180, 3.5132, 4.7421$	12.0426
88	9	$01^2123456$	$0, 0.1289, 0.5540, 1, 1.2613, 2.1326, 3, 3.6881, 4.2350$	11.8899

Table 3.

#	n	trees	Laplacian spectrum	$LE(T_n)$
89	9	011231234	0, 0.1506, 0.4266, 1, 1.4229, 2.1724, 3, 3.4576, 4.3699	11.7776
90	9	011212345	0, 0.1404, 0.5362, 0.7754, 1.5803, 2.2449, 2.7784, 3.5988, 4.3455	11.7131
91	9	$0(123)^2 12$	0, 0.1981, 0.3004, 1, 1.5550, 2.2391, 3, 3.2470, 4.4605	11.6709
92	9	$0(12)^2 1234$	0, 0.1708, 0.3820, 0.8503, 1.6761, 2.4165, 2.6180, 3.4421, 4.4442	11.6194
93	9	$0(1234)^2$	0, 0.1206, 0.4679, 1, 1.6527, 2.3473, 3, 3.5321, 3.8794	11.2954
94	10	01^9	0, 1^8, 10	16.4000
95	10	$01^7 12$	0, 0.4355, 1^6, 2.5464, 9.0181	15.9290
96	10	$01^6 12^2$	0, 0.3451, 1^6, 3.5956, 8.0593	16.1098
97	10	$01^6 123$	0, 0.2560, 1^5, 1.5227, 3.1964, 8.0249	15.2426
98	10	$01^5 (12)^2$	0, 0.3820, 0.5085, 1^4, 2.4434, 2.6180, 8.0480	15.4189
99	10	$01^5 12^3$	0, 0.3087, 1^6, 4.5111, 7.1801	16.1825
100	10	$01^4 12^4$	0, 0.2984, 1^6, 5, 6.7016	16.2032
101	10	$01^5 123^2$	0, 0.2076, 1^5, 1.6797, 4.0748, 7.0378	15.0253
102	10	$01^5 1223$	0, 0.2535, 0.6693, 1^4, 2.1879, 3.7996, 7.0897	15.3544
103	10	$01^4 1212^2$	0, 0.3130, 0.5041, 1^4, 2.4928, 3.5709, 7.1192	15.5658
104	10	$01^5 1234$	0, 0.1775, 1^5, 2.2914, 3.495621, 7.0355	14.8450
105	10	$01^4 12123$	0, 0.2436, 0.4813, 1^3, 1.5090, 2.5055, 3.1921, 7.0686	14.7115
106	10	$01^3 (12)^3$	0, 0.3820^2, 0.6168, 1^2 2.2835, 2.6180^2, 7.0996	14.8383
107	10	$01^4 123^3$	0, 0.1902, 1^5, 1.7292, 5, 6.0806	14.9612
108	10	$01^4 12^2 23$	0, 0.2592, 0.5749, 1^4, 2.3791, 4.4541, 6.3327	15.5318
109	10	$01^3 1212^3$	0, 0.2767, 0.5283, 1^4, 2.4558, 4.3827, 6.3565	15.5900
110	10	$01^4 1223^2$	0, 0.2183, 0.6439, 1^4, 2.6982, 4.2803, 6.1592	15.4755
111	10	$01^3 (12^2)^2$	0, 0.2679, 0.4921, 1^4, 3.2444, 3.7321, 6.2635	15.6800
112	10	$01^4 1234^2$	0, 0.1512, 1^5, 2.6112, 4.1820, 6.0555	14.8975
113	10	$01(12^3)^2$	0, 0.2087, 0.6385, 1^4, 2.8326, 4.7913, 5.5289	15.5056
114	10	$01^4 12334$	0, 0.1723, 0.6127, 1^4, 2.3922, 4.2090, 6.0621	15.0783
115	10	$01^2 12334$	0, 0.1967, 0.5011, 1^3, 1.6429, 2.4895, 4.0600, 6.1098	14.5186
116	10	$01^3 12^2 23^2$	0, 0.2370, 0.5379, 1^4, 3.1628, 4.3148, 5.7475	15.6502
117	10	$01^4 12234$	0, 0.1875, 0.5604, 1^3, 1.6972, 1.4131, 2.8596, 3.8315, 6.1478	14.8779
118	10	$01^4 1(23)^2$	0, 0.2442, 0.3820, 1^3, 1.7101, 2.6180, 3.8920, 6.1537	14.5274
119	10	$012^2 1^3 123$	0, 0.2222, 0.44121, 1^3, 1.4963, 3.1080, 3.5435, 6.1888	14.8806
120	10	$01^3 121223$	0, 0.2357, 0.5027, 0.6782, 1^2, 2.1648, 2.4725, 3.7543, 6.1917	14.7667
121	10	$012^2 1^1 (12)^2$	0, 0.3030, 0.3820, 0.6596, 1^2, 2.2704, 2.6180, 3.5379, 6.2291	14.9108
122	10	$012^2 123^3$	0, 0.1442, 1^5, 2.6784, 5, 5.1774	14.9116
123	10	$012^3 12^2 23$	0, 0.1775, 0.5379, 1^3, 1.6516, 2.4544, 4.8393, 5.3393	14.4660
124	10	$01^4 12345$	0, 0.1370, 0.7110, 1^3, 1.6764, 2.7685, 3.6520, 6.0550	14.1511
125	10	$01^3 12^2 234$	0, 0.2022, 0.4696, 1^3, 1.4719, 3.0620, 4.1122, 5.6821	15.6821
126	10	$01^2 1(123)^2$	0, 0.1981, 0.3676, 1^2, 1.4469, 1.5550, 3.0787, 3.2470, 6.1068	14.0649
127	10	$01^2 1121234$	0, 0.1704, 0.4911, 1^3, 2.2293, 2.5173, 3.4880, 6.1038	14.2769

Table 4.

#	n	trees	Laplacian spectrum	$LE(T_n)$
128	10	$01^312(23)^2$	0, 0.2942, 0.3820, 0.7541, 1^2, 2.1157, 2.6180, 4.1533, 5.7127	14.7994
129	10	01^21212^223	0, 0.2330, 0.5188, 0.6158, 1^2, 2.3111, 2.4408, 4.1701, 5.7105	14.8649
130	10	$01^2(12)^2123$	0, 0.2366, 0.3820, 0.6298, 1, 1.4757, 2.3204, 2.6180, 3.1871, 6.1504	14.1518
131	10	$0(12)^3112$	0, 0.3820^3, 0.8074, 2, 2.6180^3, 6.1926	14.0932
132	10	012^31223^2	0, 0.1600, 0.5669, 1^3, 1.5488, 3.0612, 4.5184, 5.1446	14.6485
133	10	0112^3123^2	0, 0.1640, 0.5531, 1^3, 1.5129, 3.2827, 4.1972, 5.2902	14.7401
134	10	$01^212^2123^2$	0, 0.1800, 0.4792, 1^3, 1.6050, 3.3220, 4.0680, 5.3458	14.6716
135	10	012^31212^2	0, 0.2357, 0.3380, 1^3, 1.8265, 3.0847, 4.1617, 5.3534	14.4526
136	10	0112^31223	0, 0.1881, 0.5415, 0.7203, 1^2, 2.0898, 2.8800, 4.2424, 5.3379	14.7002
137	10	$01^2121223^2$	0, 0.2015, 0.5188, 0.6721, 1^2, 2.3111, 2.7424, 4.1701, 5.3839	14.8151
138	10	01^212^21223	0, 0.2100, 0.4862, 0.6872, 1^2, 2.1532, 3.1369, 3.8639, 5.4627	14.8333
139	10	$0112(12^2)^2$	0, 0.2679, 0.3426, 0.7482, 1^2, 2.2345, 3.1746, 3.7321, 5.5002	14.8838
140	10	$0(123^2)^2$	0, 0.1231, 0.6842, 1^3, 1.7846, 3.0971, 4.2124, 5.0986	14.0162
141	10	012^312234	0, 0.1466, 0.4582, 1^2, 1.3494, 1.6947, 3.0251, 4.1959, 5.1302	13.9023
142	10	012^312334	0, 0.1353, 0.5723, 1^3, 2.1398, 2.7545, 4.2955, 5.1026	14.1848
143	10	012^212^2234	0, 0.1648, 0.3883, 1^2, 1.4358, 1.6684, 3.1150, 4.0226, 5.2050	13.8853
144	10	$012^31(23)^2$	0, 0.1695, 0.3820, 1^2, 1.2228, 2.1971, 2.6180, 4.2756, 5.1350	14.0514
145	10	012^2123^234	0, 0.1442, 0.5188, 1^3, 2.3111, 2.6784, 4.1701, 5.1774	14.2740
146	10	$0(12^2)^3$	0, 0.2679^2, 1^3, 2, 3.7321^2, 5	14.5284
147	10	$01^3122345$	0, 0.1473, 0.5085, 1^3, 2.1053, 3.1771, 3.7773, 5.2844	14.2883
148	10	0112^21223^2	0, 0.1792, 0.5188, 0.7141, 1^2, 2.3111, 3.1593, 4.1701, 4.9474	14.7158
149	10	$012^2231223$	0, 0.1626, 0.5188, 0.6270, 1, 1.5072, 2.3111, 2.5027, 4.1701, 5.2005	14.7158
150	10	012^21^21234	0, 0.1592, 0.4563, 1^3, 2.2121, 3.2583, 3.5836, 5.3305	14.3690
151	10	$012^212(23)^2$	0, 0.1898, 0.3820, 0.7154, 1, 1.5268, 2.2743, 2.6180, 4.0296, 5.2641	13.9720
152	10	$01^21231223$	0, 0.1861, 0.4111, 0.6824, 1, 1.4697, 2.1671, 3.0584, 3.6781, 5.3472	14.1015
153	10	0112^223123	0, 0.1769, 0.4716, 0.6288, 1, 1.4112, 2.3497, 2.8697, 3.7491, 5.3430	14.2230
154	10	$012^2112123$	0, 0.2207, 0.3298, 0.7073, 1, 1.4285, 2.3268, 3.0917, 3.5074, 5.3876	14.2272
155	10	$01^2121(23)^2$	0, 0.2206, 0.3820, 0.5548, 1, 1.6771, 2.4011, 2.6180, 3.7871, 5.3595	13.9312
156	10	$011212(23)^2$	0, 0.2263, 0.3820, 0.6274, 0.7726, 2, 2.2925, 2.6180, 3.6837, 5.3975	13.9834
157	10	$012^2(12)^3$	0, 0.2971, 0.3820^2, 1, 1.7713, 2.6180, 2.6180, 3.4942, 5.4374	13.9352
158	10	$01(123^2)^2$	0, 0.1392, 0.4384, 1^2, 1.3820, 1.7459, 3.6180, 4.1149, 4.5616	13.7890
159	10	$01^3123456$	0, 0.1148, 0.5399, 1^2, 1.2709, 2.1739, 3.0636, 3.7387, 5.0981	13.7487
160	10	$01^2123445^2$	0, 0.1338, 0.5188, 1^3, 2.3111, 3.2108, 4.1701, 4.6554	14.2948
161	10	012123^212^2	0, 0.1772, 0.3300, 1^2, 1.2293, 2.2486, 3.2174, 4.0502, 4.7472	14.1269
162	10	01123^21223	0, 0.1535, 0.4616, 0.7026, 1, 1.5019, 2.1589, 3.2036, 4.0827, 4.7351	13.9607
163	10	012231223^2	0, 0.1487, 0.5188, 0.6496, 1, 1.4400, 2.3111, 3.0561, 4.1701, 4.7056	14.0858
164	10	$0(12^2)^2123$	0, 0.2154, 0.2679, 1^2, 1.2059, 2.3671, 3.3375, 3.7321, 4.8742	14.2217
165	10	$01^21231234$	0, 0.1490, 0.3621, 1^2, 1.4749, 2.2401, 3.1091, 3.4819, 5.1830	13.6281

Table 4. (*cont.*)

#	n	trees	Laplacian spectrum	$LE(T_n)$
166	10	0112^212234	0, 0.1614, 0.4439, 0.6905, 1, 1.4077, 2.4604, 3.083, 3.9006, 4.8522	14.1930
167	10	$01^21212345$	0, 0.1317, 0.5006, 0.7370, 1, 1.6424, 2.3851, 2.7880, 3.6407, 5.1744	13.5765
168	10	$012^2121223$	0, 0.2076, 0.3326, 0.6394, 1, 1.7049, 2.2883, 3.0761, 3.8552, 4.8958	13.8309
169	10	$011(23)^212^2$	0, 0.1920, 0.3820, 0.6047, 1, 1.6249, 2.6180, 2.7557, 3.9407, 4.8821	13.9926
170	10	0112231223	0, 0.1729, 0.4755, 0.6617, 0.7420, 2, 2.2091, 2.9065, 3.9563, 4.8760	13.8958
171	10	$0112(123)^2$	0, 0.1981, 0.2937, 0.6603, 1.3427, 1.5550, 2.3831, 3.0686, 3.2470, 5.2516	13.5005
172	10	$01(12)^21234$	0, 0.1655, 0.3820, 0.6815, 1, 2, 2.4314, 2.6180, 3.4768, 5.2448	13.5420
173	10	$0(12)^3123$	0, 0.2318, 0.3820^2, 1.2769, 2, 2.6180^2, 3.1815, 5.3098	13.4546
174	10	0123^21234^2	0, 0.1088, 0.5188, 1^2, 1.2954, 2.3111, 3.3174, 4.1701, 4.2784	13.7540
175	10	012234512^2	0, 0.1288, 0.3924, 1^2, 1.5222, 2.2184, 3.3439, 3.9000, 4.4944	13.5133
176	10	011231234^2	0, 0.1257, 0.4097, 1^2, 1.4295, 2.4234, 3.0954, 4.0925, 4.4238	13.6702
177	10	012334123^2	0, 0.1172, 0.5188, 0.7586, 1, 1.6674, 2.3111, 3.0846, 4.1701, 4.3721	13.4759
178	10	012323412^2	0, 0.1640, 0.2885, 1^2, 1.6385, 2.3252, 3.0979, 3.9293, 4.5566	13.4180
179	10	011234512^2	0, 0.1236, 0.4790, 0.7723, 1, 1.5904, 2.5350, 3.1669, 3.6885, 4.6442	13.6693
180	10	0122341223	0, 0.1378, 0.4258, 0.6323, 1.3282, 1.5820, 2.3435 3.0242, 3.9923	13.3879
181	10	$0(123)^2123^2$	0, 0.1392, 0.3820, 0.8299, 1, 1.7459, 2.6180, 2.6889, 4.1149, 4.4812	13.4060
182	10	0122312334	0, 0.1277, 0.5188, 0.6297, 1, 2, 2.3111, 2.7968, 4.1701, 4.4458	13.4476
183	10	$0(123)^212^2$	0, 0.1981, 0.2375, 1^2, 1.5550, 2.5634, 3.2470, 3.4832, 4.7159	13.6189
184	10	0112312234	0, 0.1487, 0.3820, 0.6496, 1.3820, 1.4400, 2.6180, 3.0561, 3.6180, 4.7056	13.5954
185	10	012123412^2	0, 0.1566, 0.3280, 0.8452, 1, 1.7534, 2.4520, 3.1820, 3.5756, 4.7070	13.4334
186	10	0122334123	0, 0.1398, 0.4249, 0.6932, 1, 2, 2.2574, 3.1456, 3.6414, 4.6978	13.4843
187	10	$01(23)^21223$	0, 0.1561, 0.3820, 0.5965, 1.1864, 2, 2.4539, 2.6180, 4.0305, 4.5767	13.3581
188	10	$0(12)^212234$	0, 0.1700, 0.3820, 0.5078, 1.3820, 1.6959, 2.6180, 2.8758, 3.6180, 4.7505	13.3246
189	10	0121231223	0, 0.1859, 0.2989, 0.6329, 1.1826, 2, 2.3183, 3.0437, 3.5861, 4.7517	13.3995
190	10	$0(12)^21(23)^2$	0, 0.2087, 0.3820^2, 2, 2.6180, 2.6180, 3.6180, 4.7913	13.2906
191	10	$01^21234567$	0, 0.1029, 0.4363, 1^2, 1.7250, 2.5064, 3.2255, 3.7678, 4.2357	13.0708
192	10	0112312345	0, 0.1172, 0.3820, 0.7586, 1.3820, 1.6674, 2.6180, 3.0846, 3.6180, 4.3721	12.9855
193	10	$01(1234)^2$	0, 0.1206, 0.3489, 1^2, 2, 2.3473, 3.2739, 3.5321, 4.3772	13.0610
194	10	0112123456	0, 0.1100, 0.4616, 0.6697, 1.2415, 2, 2.4010, 3.0579, 3.7120, 4.3463	13.0344
195	10	$0(123)^3$	0, 0.1981^2, 0.8299, 1.5550^2, 2.6889, 3.2470^2, 4.4812	12.9280
196	10	0121231234	0, 0.1479, 0.2814, 0.7873, 1.2931, 2, 2.4631, 3.0926, 3.4687, 4.4659	12.9806
197	10	$0(12)^212345$	0, 0.1277, 0.3820, 0.6297, 1.3820, 2, 2.6180, 2.7968, 3.6180, 4.4458	12.9572
198	10	0123412345	0, 0.0979, 0.3820, 0.8244, 1.3820, 2, 2.6180, 3.1756, 3.6180, 3.9021	12.6274

□

3. Discussion and concluding remarks

In this paper, our attention was focused on the Laplacian energy of trees and on the partial proof of the Conjecture 1.1. We have shown that Laplacian energy of any tree T_n, with $n \leq 10$ vertices is strictly between the Laplacian energy of the path P_n and the Laplacian energy of the star S_n, a step towards the proof of the Conjecture 1.1.

Future work

let $S_{1,p}$ $(p \geq 1)$ and $S_{1,q}$ $(q \geq 1)$ are two stars, we introduced a special tree, denoted by $\mathbf{T}(p,q)$ by identifying one pendent vertex of $S_{1,p}$ and one pendent vertex of $S_{1,q}$, now we see that $\mathbf{T}(p,q)$ has diameter 4, with $n = p + q + 3$ vertices; see the graph in FIGURE 2. We plan to show that the Conjecture 1.1, is hold for any tree T_n such that $T_n \cong \mathbf{T}(p,q)$.

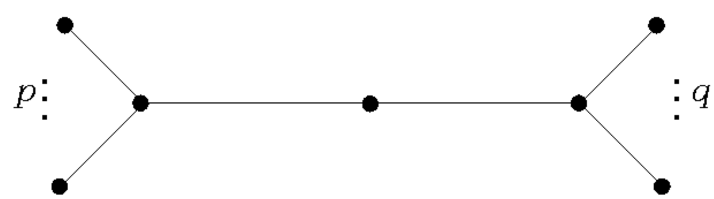

Figure 2. The graph $\mathbf{T}(p,q)$

Moreover, from [10, p. 69-83] we find all trees T_n where $n \in \{11, 12\}$, there are total 785 trees T_n for which $n \in \{11, 12\}$. We also plan to show that the Conjecture 1.1, is hold for any tree T_n where $n \in \{11, 12\}$.

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