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On the Ramsey number of 4-cycle versus wheel

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Abstract

For any fixed graphs G and H, the Ramsey number $R(G, H)$ is the smallest positive integer n such that for every graph F on n vertices must contain G or the complement of F contains H. The girth of graph G is a length of the shortest cycle. A k-regular graph with the girth g is called a (k, g)-graph. If the number of vertices in (k, g)-graph is minimized then we call this graph a (k, g)-cage. In this paper, we derive the bounds of Ramsey number $R(C_4, W_n)$ for some values of n. By modifying $(k, 5)$-graphs, for $k = 7$ or 9, we construct these corresponding (C_4, W_n)-good graphs.

Keywords: Ramsey number, good graph, order, cycle, wheel, girth
Mathematics Subject Classification: 05C55

1. Introduction

In this paper, we consider a finite undirected graphs without loops or multiple edges. Let G be graphs. The sets of vertices and edges of graph G are denoted by $V(G)$ and $E(G)$, respectively. The symbols $\delta(G)$ and $\triangle(G)$ represents the smallest and the greatest degree of vertices in G, respectively. Let C_n be a cycle with n vertices and W_n be a wheel on n vertices obtained from a C_{n-1} by adding one vertex x and making x adjacent to all vertices of the C_{n-1}. The *girth* of a graph G is the length of its shortest cycle in G. A k-regular graph with girth g is called a (k, g)-*graph*. A (k, g)-graph with minimum number of vertices is called a (k, g)-cage. For fixed graphs G and H, a graph F is called a (G, H)-*good graph* if F contains no G and F complement contains no

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H. Any (G, H)-good graph with n vertices will be called a (G, H, n)-*good graph*. The ***Ramsey number*** $R(G, H)$ is the smallest positive integer n such that for every graph F of order n contains G or the complement of F contain H. So, the Ramsey number $R(G, H)$ is the smallest positive integer n such that there exists no (G, H, n)-good graph.

It is known that $R(C_4, W_4) = 10, R(C_4, W_5) = 9$ and $R(C_4, W_6) = 10$ (cf.[2]). Tse [2] determined the value of $R(C_4, W_m)$ for $7 \leq m \leq 13$. Dybizbański dan Dzido [2] determined that $R(C_4, W_m) = m + 4$ for $14 \leq m \leq 16$ and $R(C_4, W_{q^2+1}) = q^2 + q + 1$ for prime power $q \geq 4$. Recently, Zhang, Broersma and Chen [2] show that $R(C_4, W_n) = R(C_4, S_n)$ for $n \geq 7$. Based on this result and Parsons' results on $R(C_4, S_n)$, they derived the best possible general upper bound for $R(C_4, W_n)$ and determined some exact values of them. In general, the exact value of the Ramsey number $R(C_4, W_n)$ is still open for $n \geq 17$ with the exception for several values of n. In this paper, we derive the bounds of Ramsey number $R(C_4, W_n)$ for some values of n. By modifying $(k, 5)$-graphs, for $k = 7$ or 9, we construct these corresponding (C_4, W_n)-good graphs.

Theorem 1.1. *Each of the following statements must hold.*

(i) *For any $m \geq 18$ there exists a graph G of order m with $\delta(G) = 4$ and $G \not\supseteq C_4$.*

(ii) *For any even $m \geq 50$ there exists a graph G of order m with $\delta(G) = 5$ and $G \not\supseteq C_4$.*

(iii) $R(C_4, W_{2k+1}) \geq R(C_4, W_{2k})$ *for any* $k \geq 25$.

(iv) $R(C_4, W_{m+n}) \geq \max\{R(C_4, W_m), R(C_4, W_n)\}$ *with* $\min\{m, n\} \geq 7$ *and* $\max\{m, n\} \geq 50$.

Theorem 1.2. *The upper and lower bounds of the Ramsey number $R(C_4, W_m)$ for any $m \in [46, 93]$ are as follows.*

(i) $m + 6 \leq R(C_4, W_m) \leq m + 7$, *for* $46 \leq m \leq 51$.

(ii) $m + 8 \leq R(C_4, W_m) \leq m + 9$, *for* $79 \leq m \leq 82$,

(iii) $m + 8 \leq R(C_4, W_m) \leq m + 10$, *for* $83 \leq m \leq 87$.

(iv) $97 \leq R(C_4, W_{88}) \leq 98$ *and* $m + 8 \leq R(C_4, W_m) \leq m + 10$, *for* $89 \leq m \leq 93$.

2. Proofs of the main results

To prove Theorems 1.1 and 1.2, we need the following two lemmas and one theorem.

Lemma 2.1. *[2] If G is a (C_4, W_m, n)-*good graph *for $7 \leq m \leq n - 4$ then $\delta(G) \geq n - m + 1$.*

Lemma 2.2. *[2] If G contains no C_4 with n vertices and $\delta(G) = d$ then $d^2 - d + 1 \leq n$.*

Theorem 2.1. *[2] For all integers $m \geq 11$, $R(C_4, W_m) \leq m + \lfloor\sqrt{m - 2}\rfloor + 1$.*

Proof Theorem 1.1.

(i) For any integer $m \geq 18$, construct a graph G on m vertices with $\delta(G) = 4$ and $G \not\supseteq C_4$ by considering the following two cases.

(a) **Case 1** $m = 2k, k \geq 9$.

First, if $k \neq 12$ define the vertex-set and edge-set of G as follows.

- $V(G) = \{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$, and
- $E(G) = \{a_i b_i, b_i a_{i+1}, a_i a_{i+1}, b_i b_{i+3} : 1 \leq i \leq k$ and all indices are in mod $k\}$.

Note that all indices are calculated in mod k. It is clear that vertex a_i is adjacent to each of $\{b_i, b_{i-1}, a_{i+1}, a_{i-1}\}$ and b_i is adjacent to each of $\{a_i, a_{i+1}, b_{i+3}, b_{i-3}\}$ for all $i = 1, 2, \cdots, k$. Thus, $\delta(G) = 4$. Now, we will show that $G \not\supseteq C_4$. For a contradiction, suppose G contains a C_4. Since $k \neq 12$, the four vertices of C_4 cannot be all b_i. Therefore, this C_4 must contain at least one vertex a_i. Now, consider the following 3 subcases.

- Subcase 1. $a_i b_i \in C_4$ for some i. If a_i and b_i are the first and second vertices of this C_4 then the possible third and fourth vertices are listed in Table 1. However, we have that no vertex 4 is adjacent to vertex 1. Therefore, there is no such C_4 occur. Thus, $a_i b_i$ is not an edge in C_4.

vertex 1	vertex 2	vertex 3	vertex 4
a_i	b_i	a_{i+1}	b_{i+1}
			a_i
			a_{i+2}
		b_{i+3}	a_{i+3}
			a_{i+4}
			b_{i+6}
		b_{i-3}	a_{i-3}
			a_{i-2}
			a_{i-6}

Table 1. List of possible vertices of a C_4 in Subcase 1.

- Subcase 2. $b_i a_{i+1} \in C_4$ for some i. If b_i and a_{i+1} are the first and second vertices in this C_4, and no edge $a_i b_i \in C_4$, for each i, then the possible third and fourth vertices are presented in Table 2. Clearly, each of the possible fourth vertices is not adjacent to vertex 1. Therefore, no C_4 is formed in this case.

vertex 1	vertex 2	vertex 3	vertex 4
b_i	a_{i+1}	a_{i+2}	b_{i+1}
			a_{i+3}
		a_i	b_{i-1}
			a_{i-1}

Table 2. List of possible vertices of a C_4 in Subcase 2.

- Subcase 3. $a_i a_{i+1}$ or $b_i b_{i+3} \in C_4$ for some i. From the previous subcases, we know that the edges $a_i b_i$ or $b_i a_{i+1}$ cannot be in this C_4. So, this C_4 only consist of edges $a_i a_{i+1}$ and/or $b_i b_{i+3}$ for some i. Since $k \neq 12$ then no C_4 occurs in this case.

Therefore, if $m = 2k, k \geq 9$ and $k \neq 12$ then the above graph G has m vertices with $\delta(G) = 4$ and $G \not\supseteq C_4$.

Second, for $k = 12$, consider graph G of order 24 in Figure 1. It can be verified that G containing no C_4 and $\delta(G) = 4$.

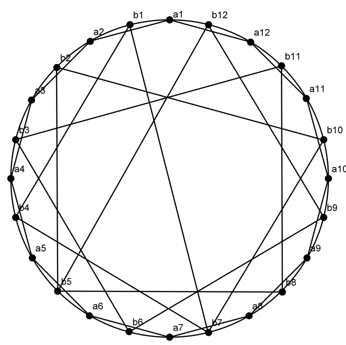

Figure 1. A graph G of order 24 containing no C_4 with $\delta(G) = 4$.

(b) **Case 2** $m = 2k + 1, k \geq 9, k \neq 11$.
In this case, if $k \neq 11$ define the vertex-set and edge-set of G as follows.
$V(G) = \{c, a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$, and
$E(G) = \{a_i b_i \mid i \in [1, k]\} \cup \{b_i a_{i+1}, a_i a_{i+1} \mid i \in [1, k-1]\} \cup \{b_i b_{i+3} \mid i \in [1, k-3]\} \cup \{b_1 b_{k-1}, b_2 b_k, a_1 b_{k-2}, ca_1, ca_k, cb_3, cb_k\}$

Note that all indices are calculated in mod k. It is easy to see that each vertex is adjacent to at least four vertices, so $\delta(G) = 4$. Now, we will show that $G \not\supseteq C_4$. For a contradiction, suppose G contains a C_4. Since $k \neq 11$, this C_4 cannot consist of vertices b_i only. Therefore, this C_4 must contain at least one vertex a_i. Now, consider the following 4 subcases.

- Subcase 1. $a_i b_i \in C_4$ for some i. If a_i, b_i are the first and second vertices in C_4 then the possible third and fourth vertices are listed in Table 3. However, there is no vertex 4 is connected to vertex 1. Therefore, this C_4 cannot contain an edge $a_i b_i$, for some i.
- Subcase 2. $b_i a_{i+1} \in C_4$ for some i or $cb_k \in C_4$. From the above subcase, this C_4 cannot contain an edge $a_i b_i$, for some i. If b_i and a_{i+1}, c are the first and second vertices in C_4 then the possible third and fourth vertices are presented in Table 4. Again, however, no vertex 4 is connected to vertex 1. Therefore, $b_i a_{i+1}$ or cb_k cannot be in C_4, for some i.
- Subcase 3. $a_i a_{i+1}, ca_k$, or $ca_1 \in C_4$, for some i. In this case, the possible vertices of this C_4 can be seen in Table 5. But, no vertex 4 is adjacent to vertex 1. Therefore, there is no such C_4 formed in this case.
- Subcase 4. $b_1 b_{k-1}, b_2 b_k, a_1 b_{k-2}, cb_3$ or $b_i b_{i+3} \in C_4$ for some i. We can assume that b_i is the first vertex of a C_4. Then, the possible vertex of the C_4 are presented in Table 6. In this case, it is clear that no C_4 can be formed. Thus, $C_4 \not\subseteq G$.

vertex 1	vertex 2	vertex 3	vertex 4
a_i	b_i	a_{i+1}	b_{i+1}
			a_i $(i \leq k-1)$
			c $(i+1=k)$
			a_{i+2} $(i+1 \leq k-1)$
		c $(i=k)$	b_3
			a_k
			a_1
			b_k
		b_{i+3} $(1 \leq i \leq k-3)$	b_1 $(i=k-4)$
			a_{i+3}
			a_{i+4} $(i+3 \leq k-1)$
			c $(i+3=k)$
			b_{i+6} $(i+3 \leq k-6)$
			a_1 $(i+2=k-2)$
			b_2 $(i+3=k)$
		b_{k-1} $(i=1)$	a_{k-1}
			a_k
			b_{k-4}
		b_k $(i=2)$	c
			a_k
			b_{k-3}
		c $(i=3)$	a_1
			b_k
			a_k
			b_k
		a_1 $(i=k-2)$	b_1
			a_2
			c
			b_{k-2}

Table 3. List of possible vertices of a C_4 in Subcase 1.

vertex 1	vertex 2	vertex 3	vertex 4
b_i	a_{i+1} $(i \leq k-1)$	a_{i+2} $(i+1 \leq k-1)$	b_{i+1} $(i+1 \leq k-1)$
			a_{i+3} $(i+2 \leq k-1)$
			c $(i+2=k)$
			c $(i=k-1)$
			b_{k-2} $(i=k-1)$
		c $(i+1=k)$	b_k
			a_1
			b_3
			a_k
		a_i	c $(i=1)$
			b_{k-2} $(i=1)$
			b_{i-2}
			b_{i-2}
	c $(i=k)$	a_k	a_{k-1}
			b_{k-1}
		a_1	a_2
			b_{k-2}
		b_3	b_4
			b_6

Table 4. List of possible vertices of a C_4 for Subcase 2.

vertex 1	vertex 2	vertex 3	vertex 4
a_i	a_{i+1} $(i \leq k-1)$	a_{i+2} $(i+1 \leq k-1)$	a_{i+3} $(i+1 \leq k-1)$ $(i+2 \leq k-1)$
			c $(i+2=k)$
		c $(i+1=k)$	a_1
			a_k
			b_3
	c $(i=1)$	a_k	a_{k-1}
		b_3	b_6
	c $(i=k)$	a_1	a_2
			b_{k-2}
		b_3	b_6

Table 5. List of possible vertices of a C_4 in Subcase 3.

vertex 1	vertex 2	vertex 3	vertex 4
b_i	b_{i+3}	b_{i+6}	b_{i+9}
			b_1 $(i+6=k-1)$
			b_1 $(i+6=k-1)$
			b_2 $(i+6=k)$
		b_1 $(i+3=k-1)$	b_4
		a_1 $(i+3=k-2)$	
		b_2 $(i+3=k)$	b_5
	b_{k-1} $(i=1)$	b_{k-4}	b_{k-7}
	a_1 $(i=k-2)$		
	c $(i=3)$	b_3	b_6
	b_k $(i=2)$	b_{k-3}	b_{k-6}

Table 6. List of possible vertices of a C_4 in Subcase 4.

For $k=11$, we construct a graph G containing no C_4 on 23 vertices with $\delta(G)=4$ as depicted in Figure 2.

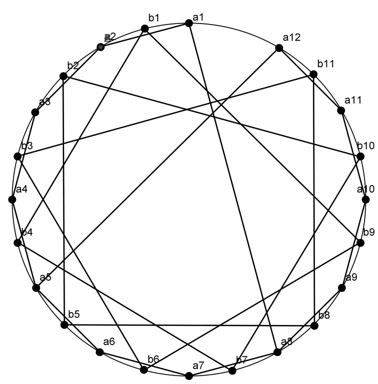

Figure 2. A graph G containing no C_4 on 23 vertices with $\delta(G)=4$.

(ii) For any even $m \geq 50$, we shall construct a graph G on m vertices with $\delta(G)=5$ and $G \not\supseteq C_4$. Let us define the vertex-set and edge-set of G:
$V(G)=\{a_1,a_2,\ldots,a_m\}$ and
$E(G)=\{a_ia_{i+1} \mid i \in [1,m]\} \cup \{a_ia_{i+4} \mid i \text{ odd}\} \cup \{a_ia_{i+12} \mid i \text{ even}\}$
$\cup \{a_ia_{i+8} \mid i=2,4,6,8, \text{ and } i=16k, \text{ for } k \in [1,\lfloor m/16 \rfloor]\}$
$\cup \{a_ia_{i+16} \mid i=1,3,5,\cdots,15, \text{ and } i=16k, \ for \ k \in [1,\lfloor m/34 \rfloor]\}$.
It can be verified easily that $\delta(G)=5$. Now, suppose that $C_4 \subseteq G$. Let C_4 be $(a_{i_1},a_{i_2},a_{i_3},a_{i_4})$ with $i_2=i_1+x_1, i_3=i_2+x_2, i_4=i_3+x_3, i_1=i_4+x_4 \mod m$. Clearly, $G \supseteq C_4$ if and only if m divides $x_1+x_2+x_3+x_4$. So, $x_1+x_2+x_3+x_4=0$ or $x_1+x_2+x_3+x_4$ is a multiple of 4. Observe that the maximum value of $x_1+x_2+x_3+x_4=48$ which is achieved when i is even. It is easy to see that $x_1+x_2+x_3+x_4 \neq 0$. Therefore, m never divides $(x_1+x_2+x_3+x_4)$. Thus, $G \not\supseteq C_4$.

(iii) We will show that $R(C_4, W_{2k+1}) \leq R(C_4, W_{2k})$ for any $k \geq 25$. By Theorem 1.1(ii), we have a graph G on $m = 2k + 4 \geq 50$ vertices with $\delta(G) \leq 5$ and $G \nsubseteq C_4$. Then, $\triangle(\overline{G}) \leq 2k - 2$. Thus,$\overline{G} \nsupseteq W_{2k}$. Therefore, we obtain that G is a $(C_4, W_{2k}, 2k + 4)$-*good graph*. As a consequence, $R(C_4, W_{2k}) \leq 2k+5$. Now, let $R(C_4, W_{2k}) = m$. By Lemma 2.1, there exists a $C_4, W_{2k}, m - 1$-good graph G with $\delta(G) \geq m - 1 - 2k + 1 = m - 2k$. Thus, $\triangle(\overline{G}) \leq (m-1)-(m-2k) = 2k-1$. This means that G is also a $(C_4, W_{2k+1}, m+1)$-good graph. Therefore $R(C_4, W_{2k+1}) \leq R(C_4, W_{2k})$.

(iv) We will show that $R(C_4, W_{m+n}) \geq \max\{R(C_4, W_m), R(C_4, W_n)\}$ with $\min\{m, n\} \geq 7$ and $\max\{m, n\} \geq 50$. Without lost of generality, let $R(C_4, W_m) = \max\{R(C_4, W_m), R(C_4, W_n)\}$. If m is even, by Theorem 1.1(ii) there exists graph G on $m + 4$ with $\delta(G) = 5$ and $C_4 \nsubseteq G$. Then, $\triangle(\overline{G}) \leq m - 2$. Then, $W_m \nsubseteq \overline{G}$. Therefore, we obtain that G is a $(C_4, W_m, m + 4)$-*good graph*. As a consequence, $R(C_4, W_m) \geq m + 5$ and by Theorem 1.1(3), we have $R(C_4, W_m) \geq m + 5$ for all $m \geq 50$. Now, let $R(C_4, W_m) = p$. By Lemma 2.1, there exists a $(C_4, W_m, p - 1)$-*good graph* G with $\delta(G) \geq p - 1 - m + 1 = p - m$. Thus, $\triangle(\overline{G}) \leq (p - 1) - (p - m) = m - 1 \leq m + n - 1$. This means that G is also a $R(C_4, W_{m+n}, p - 1)$-*good graph*. Therefore,$R(C_4, W_{m+n}) \geq \max\{R(C_4, W_m)$.

□

Proof Theorem 1.2.

(i) We will show that $m+6 \leq R(C_4, W_m) \leq m+7$, for $46 \leq m \leq 51$. Hoffman and Singleton [**??**] have constructed a $(7, 5)$-cage HS_{50} as follow. Let $V(HS_{50}) = \{a_1, a_2, \cdots, a_{50}\}$. All edges of HS_{50} are presented in Table 7.

We construct a new graph G_i on i vertices, for each $i \in [51, 56]$ as follows.

$$\begin{aligned} V(G_{51}) &= V(HS_{50}) \cup \{51\} \\ E(G_{51}) &= E(HS_{50}) \setminus \{(1,2),(2,34),(20,21),(21,22),(19,41),(34,41)\} \\ &\quad \cup\{(51,i) | i \in \{1,2,19,21,34,41\}\} \\ V(G_{52}) &= V(G_{51}) \cup \{52\} \\ E(G_{52}) &= E(G_{51}) \setminus \{(10,11),(11,12),(3,4),(3,16),(5,20)\} \\ &\quad \cup\{(52,i) | i \in \{2,3,11,12,16,20\}\} \end{aligned}$$

1	2	19	29	32	44	47	50	26	10	13	22	25	27	33	50
2	1	3	6	10	21	24	34	27	3	19	26	28	31	39	43
3	2	4	8	16	27	37	46	28	5	14	23	27	29	34	45
4	3	5	11	18	22	32	48	29	1	11	17	25	28	30	37
5	4	6	9	20	28	38	50	30	6	15	22	29	35	46	31
6	2	5	7	13	30	40	43	31	9	12	24	27	30	32	49
7	6	8	116	19	23	33	49	32	1	4	14	31	33	36	40
8	3	7	9	14	25	35	44	33	7	17	26	32	34	38	46
9	5	8	10	17	31	41	47	34	2	12	28	33	35	41	48
10	2	9	11	15	26	36	45	35	8	18	30	34	36	39	50
11	4	7	10	12	29	39	42	36	10	20	23	35	32	37	43
12	11	13	16	20	31	34	44	37	3	13	29	36	38	41	49
13	6	12	14	18	26	37	47	38	5	15	24	33	37	39	44
14	8	13	15	21	28	32	42	39	11	21	27	35	38	40	47
15	10	14	16	19	30	38	48	40	6	16	25	32	39	41	45
16	3	12	15	17	23	40	50	41	9	19	22	34	37	40	42
17	9	16	18	21	29	33	43	42	11	14	24	41	43	46	50
18	4	13	17	19	24	35	45	43	6	17	27	36	42	44	48
19	1	7	15	18	20	27	41	44	1	8	12	22	38	43	45
20	5	12	15	21	25	36	46	45	10	18	28	40	44	46	49
21	2	14	17	20	22	39	49	46	3	20	30	33	42	45	47
22	4	21	23	26	30	41	44	47	1	9	13	23	39	46	48
23	7	16	22	24	28	36	47	48	4	15	25	34	43	47	49
24	2	18	23	25	31	38	42	49	7	21	31	37	45	48	50
25	8	20	24	26	29	40	48	50	1	5	16	26	35	42	49

Table 7. The Hoffman and Singleton graph HS_{50}.

$$\begin{aligned}
V(G_{53}) &= V(G_{52}) \cup \{53\} \\
E(G_{53}) &= E(G_{52}) \setminus \{(5,9),(4,11),(31,32)\} \\
&\quad \cup \{(53,i) | i \in \{4,5,9,10,11,,31\}\} \\
V(G_{54}) &= V(G_{53}) \cup \{54\} \\
E(G_{54}) &= E(G_{53}) \setminus \{(22,30),(18,35),(30,35),(21,39)\} \\
&\quad \cup \{(54,i) | i \in \{4,21,27,30,35,39\}\} \\
V(G_{55}) &= V(G_{54}) \cup \{55\} \\
E(G_{55}) &= E(G_{54}) \setminus \{(1,50),(5,50),(32,36),(35,36)\} \\
&\quad \cup \{(55,i) | i \in \{1,5,19,35,36,50\}\} \\
V(G_{56}) &= V(G_{55}) \cup \{56\} \\
E(G_{56}) &= E(G_{55}) \setminus \{(7,23),(17,33),(16,23),(26,33)\} \\
&\quad \cup \{(56,i) | i \in \{7,16,17,22,23,33\}\}
\end{aligned}$$

Consider graph G_{51}. Clearly, $\delta(G_{51}) = 6$. Now, we will show that $C_4 \nsubseteq G_{51}$. For a contradiction, suppose $C_4 \subseteq G_{51}$. If $C_4 \subseteq G_{51}$ then this C_4 must consists of vertex 51, two vertices adjacent to 51, say x and y, and one other vertex adjacent to x and y. If vertex 51 is the first vertex of this C_4 then $\{x,y\} \subset \{a_1, a_2, a_{19}, a_{21}, a_{34}, a_{41}\}$. However, there is no other vertex adjacent to both x and y, see Figure 3. Therefore, there is no C_4 in G_{51}. Similarly, we have show that $\delta(G_i) = 6$ and $C_4 \nsubseteq G_i$ for all $i \in \{52, \cdots, 56\}$.

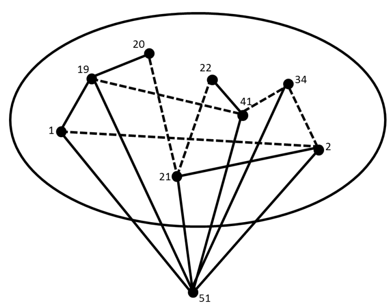

Figure 3. Possible C_4 in G_{51}.

Now, we have $\Delta(\overline{G}_i) \leq i - 7$. Thus, $W_{i-5} \nsubseteq \overline{G}_i$. As a consequence, $R(C_4, W_{i-5}) \geq i$ for all $i \in \{51, \ldots, 56\}$. By Theorem 2.1, $R(C_4, W_m) \leq m + 7$, for $46 \leq m \leq 51$. Thus, $m + 6 \leq R(C_4, W_m) \leq m + 7$ for $46 \leq m \leq 51$.

(ii) We will show that $m + 8 \leq R(C_4, W_m) \leq m + 9$ for $79 \leq m \leq 82$. From [2], there exists a $(9,5)$-graph on 96 vertices, call it G_{96}. Let $V(G_{96}) = \{0, 1, 2, \cdots, 95\}$ and all edges of graph G_{96} are presented in Table 8. We construct a graph G_i on i vertices for $86 \leq i \leq 95$, $\delta(G_i) = 8$ and $C_4 \nsubseteq G_i$. Graph G_i is obtained by removing a single vertex of G_{i+1} as follows:

$$V(G_i) = V(G_{i+1}) \setminus \{a\}$$

with a respectively $95, 79, 1, 13, 18, 36, 40, 46, 47, 63$. Now, we have $\Delta(\overline{G}_i) \leq i - 9$. Thus, $W_{i-7} \nsubseteq \overline{G}_i$. Therefore, we obtain that G_i is (C_4, W_{i-7}, i)-*good graph*. As a consequence $R(C_4, W_{i-7}) \geq i + 1$ for $79 \leq m \leq 87$ with $m = i - 7$. By Theorem 2.1, $R(C_4, W_m) \leq m + 9$, for $79 \leq m \leq 82$.

(iii) By Theorem 2.1, $R(C_4, W_m) \leq m + 10$, for $83 \leq m \leq 87$ and by the constructions in Theorem 1.2 (ii), we have $R(C_4, W_m) \geq m + 8$, for $83 \leq m \leq 87$.

(iv) We will show that $97 \leq R(C_4, W_{88}) \leq 98$ and $m + 8 \leq R(C_4, W_m) \leq m + 10$ for $89 \leq m \leq 93$. Graph G_{96} is $(9, 5)$-graph. Thus, $\Delta(\overline{G}_{96}) = 96 - 1 - 9 = 86$. Therefore, we obtain that G_{96} is a $(C_4, G_{88}, 96)$-*good graph* and G_{96} is a $(C_4, G_{89}, 96)$-*good graph*. As a consequence $R(C_4, G_{88}) \geq 97$ and $R(C_4, G_{89}) \geq 97$. For $90 \leq m \leq 93$, we construct graph G_i on i vertices, with $97 \leq i \leq 100$ as follows.

$$
\begin{aligned}
V(G_{97}) &= V(G_{96}) \cup \{96\} \\
E(G_{97}) &= E(G_{96}) \setminus \{(8,16), (53,77), (16,93), (0,8), (0,77), (34,82), (5,53), (24,58)\} \\
&\quad \cup \{(96,i) | i \in \{0, 8, 16, 24, 53, 77, 93, 82\}\} \\
V(G_{98}) &= V(G_{97}) \cup \{97\} \\
E(G_{98}) &= E(G_{97}) \cup \{(97,i) | i \in \{34, 26, 18, 10, 58, 5, 87, 64\}\} \\
&\quad \setminus \{(18,26), (10,18), (60,26), (5,64), (10,87), (58,82), (63,87), (9,64), (34,80)\} \\
V(G_{99}) &= V(G_{98}) \cup \{98\} \\
E(G_{99}) &= E(G_{98}) \cup \{(98,i) | i \in \{3, 6, 11, 19, 48, 65, 80, 88\}\} \\
&\quad \setminus \{(3,11), (31,65), (11,19), (6,46), (3,52), (6,88), (19,65), (56,80), (72,88), \\
&\quad (14,48)\} \\
V(G_{100}) &= V(G_{99}) \cup \{99\} \\
E(G_{100}) &= E(G_{99}) \cup \{(99,i) | i \in \{1, 7, 25, 33, 41, 43, 52, 81\}\} \\
&\quad \setminus \{(33,41), (1,41), (1,50), (43,89), (33,92), (25,54), (7,89), (7,62), (25,84), \\
&\quad (4,52), (35,81)\}
\end{aligned}
$$

From the construction, we have $\Delta(\overline{G}_i) = i - 9$ for $97 \leq i \leq 100$. Thus, $W_{i-7} \nsubseteq \overline{G}_i$. Therefore, we obtain that G_i is (C_4, W_{i-7}, i)-*good graph*. As consequence $R(C_4, W_m) \geq m + 8$ for $90 \leq m \leq 93$ with $m = i - 7$. By Theorem 2.1, $R(C_4, W_m) \leq m + 10$ for $88 \leq m \leq 93$.

□

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0	8	40	48	49	55	59	77	82	94	48	0	2	14	19	37	41	47	64	72
1	9	41	49	50	56	60	78	83	95	49	0	1	3	15	20	38	42	65	73
2	10	42	48	50	51	57	61	79	84	50	1	2	4	16	21	39	43	66	74
3	11	43	49	51	52	58	62	80	85	51	2	3	5	17	22	40	44	67	75
4	12	44	50	52	53	59	63	81	86	52	3	4	6	18	23	41	45	68	76
5	13	45	51	53	54	60	64	82	87	53	4	5	7	19	24	42	46	69	77
6	14	46	52	54	55	61	65	83	88	54	5	6	8	20	25	43	47	70	78
7	15	47	53	55	56	62	66	84	89	55	0	6	7	9	21	26	44	71	79
8	0	16	54	56	57	63	67	85	90	56	1	7	8	10	22	27	45	80	88
9	1	17	55	57	58	64	68	86	91	57	2	8	9	11	23	28	46	81	89
10	2	18	56	58	59	65	69	87	92	58	3	9	10	12	24	29	47	82	90
11	3	19	57	59	60	66	70	88	93	59	0	4	10	11	13	25	30	83	91
12	4	20	58	60	61	67	71	89	94	60	1	5	11	12	14	26	31	84	92
13	5	21	59	61	62	68	72	90	95	61	2	6	12	13	15	27	32	85	93
14	6	22	48	60	62	63	69	73	91	62	3	7	13	14	16	28	33	86	94
15	7	23	49	61	63	64	70	74	92	63	4	8	14	15	17	29	34	87	95
16	8	24	50	62	64	65	71	75	93	64	5	9	15	16	18	30	35	48	80
17	9	25	51	63	65	66	72	76	94	65	6	10	16	17	19	31	36	49	81
18	10	26	52	64	66	67	73	77	95	66	7	11	17	18	20	32	37	50	82
19	11	27	48	53	65	67	68	74	78	67	8	12	18	19	21	33	38	51	83
20	12	28	49	54	66	68	69	75	79	68	9	13	19	20	22	34	39	52	84
21	13	29	50	55	67	69	70	76	80	69	10	14	20	21	23	35	40	53	85
22	14	30	51	56	68	70	71	77	81	70	11	15	21	22	24	36	41	54	86
23	15	31	52	57	69	71	72	78	82	71	12	16	22	23	25	37	42	55	87
24	16	32	53	58	70	72	73	79	83	72	13	17	23	24	26	38	43	48	88
25	17	33	54	59	71	73	74	80	84	73	14	18	24	25	27	39	44	49	89
26	18	34	55	60	72	74	75	81	85	74	15	19	25	26	28	40	45	50	90
27	19	35	56	61	73	75	76	82	86	75	16	20	26	27	29	41	46	51	91
28	20	36	57	62	74	76	77	83	87	76	17	21	27	28	30	42	47	52	92
29	21	37	58	63	75	77	78	84	88	77	0	18	22	28	29	31	43	53	93
30	22	38	59	64	76	78	79	85	89	78	1	19	23	29	30	32	44	54	94
31	23	39	60	65	77	79	80	86	90	79	2	20	24	30	31	33	45	55	95
32	24	40	61	66	78	80	81	87	91	80	3	21	25	31	32	34	46	56	64
33	25	41	62	67	79	81	82	88	92	81	4	22	26	32	33	35	47	57	65
34	26	42	63	68	80	82	83	89	93	82	0	5	23	27	33	34	36	58	66
35	27	43	64	69	81	83	84	90	94	83	1	6	24	28	34	35	37	59	67
36	28	44	65	70	82	84	85	91	95	84	2	7	25	29	35	36	38	60	68
37	29	45	48	66	71	83	85	86	92	85	3	8	26	30	36	37	39	61	69
38	30	46	49	67	72	84	86	87	93	86	4	9	27	31	37	38	40	62	70
39	31	47	50	68	73	85	87	88	94	87	5	10	28	32	38	39	41	63	71
40	0	32	51	69	74	86	88	89	95	88	6	11	29	33	39	40	42	56	72
41	1	33	48	52	70	75	87	89	90	89	7	12	30	34	40	41	43	57	73
42	2	34	49	53	71	76	88	90	91	90	8	13	31	35	41	42	44	58	74
43	3	35	50	54	72	77	89	91	92	91	9	14	32	36	42	43	45	59	75
44	4	36	51	55	73	78	90	92	93	92	10	15	33	37	43	44	46	60	76
45	5	37	52	56	74	79	91	93	94	93	11	16	34	38	44	45	47	61	77
46	6	38	53	57	75	80	92	94	95	94	0	12	17	35	39	45	46	62	78
47	7	39	48	54	58	76	81	93	95	95	1	13	18	36	40	46	47	63	79

Table 8. The graph G_{96}.

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