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Unsteady MHD free convective flow past a vertical plate: An automated solution approach

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Abstract The case of unsteady two-dimensional laminar free convection flow over a vertical plate by an incompressible viscous fluid is analysed in the presence of uniform magnetic field perpendicular to the flow. The governing equations in a vector form are transformed into non-dimensional form. Then, the dimensionless equations are solved using automated solution technique which is FEniCS. The effects of magnetic parameter on the velocity and temperature profiles are obtained and discussed in this paper.
Keywords Unsteady state, MHD, Vertical plate, FEniCS

INTRODUCTION

The study of magnetic field effects on the behaviour of fluid flow is called magnetohydro-dynamics (MHD). The fluid must be electrically conducting fluid with the presence of magnetic field. It is more interesting to explore MHD in convective flow due to the fact that increasing the magnetic parameter may decrease the velocity and temperature profile. For example, Soundalgekar et al. (1979) solved MHD free convective flow and found that increasing magnetic parameter lead to increase the velocity in the heated plate while decrease the velocity in the cooled plate. Pop et al. (1994) concentrated on MHD forced convective over a semi-infinite flat plate and concluded that the increasing of magnetic parameter will decrease the temperature profile. Aydın & Kaya (2008) studied the magnetic effects on mixed convection flow and summarized that increasing magnetic parameter will increase the temperature and velocity gradient at plate.

In this study, we focus on unsteady laminar case where the flow is dependent on time. Several papers studied on unsteady MHD free convective flow at vertical plate. Hossain & Mandal (1985) solved MHD at different temperature accelerated porous plate. They found that when magnetic parameter increases, the velocity decreases for greater cooling plate while velocity increases for greater heating plate. Helmy (1998) concluded that velocity decreases when magnetic parameter increases past a vertical porous plate using. In addition, same result as Helmy (1998) is presented by Abd El-Naby et al. (2003) in their paper with variable surface temperature at vertical plate.

There are papers solved time dependent convective flow using different method such as shooting method (Pop et al., 1980), Keller box method (Kumari et al., 1996) and finite difference method (Abd El-Naby et al. , 2003) . In this paper, we will use finite element method (FEM). FEM is chosen to solve fluid dynamics problems due to its capability to deal with complex geometries. The incompressible Navier–Stokes equations can be discretized in many ways i.e. Chorin's projection scheme (Chorin, 1968), the incremental pressure correction scheme (IPCS) (Goda, 1979), consistent splitting scheme (CSS) (Guermond et al., 2006) and least square galerkin stabilized method (G2) (Hoffman & Johnson, 2007). There are papers that solved time dependent convective heat transfer using discretization scheme. Wong (2007) solved mixed convection in lid driven cavity using CSS. Later, Jia et al. (2011) investigated same problem as Wong (2007) using operator splitting scheme. Meanwhile, Amine et al. (2013) implemented Chorin's method

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in solving free convection in vertical open-ended channel. Górecki & Szumbarski (2014) used unconditionally stable splitting scheme to solve free convection in square cavity.

Furthermore, we will solve our governing equations using automated solution technique which is FEniCS (Logg et al., 2012). FEniCS is an open source software project to solve partial differential equations by implementing finite element method efficiently. FEniCS can solve vector form equation directly using Python programming. Several papers started using FEniCS to solve convective heat transfer such as Zhang et al. (2016) for steady case using coupling method and Stepanov et al. (2016) for unsteady case using G2 method and streamline upwinding Petrov-Galerkin (SUPG) method.

The current work is applying automated solution approach for solving time dependent MHD free convective flow of viscous incompressible electrically conducting fluid past a vertical plate. For the discretization, we use CSS method. The magnetic field effects on the velocity and temperature profiles are discussed.

MATHEMATICAL FORMULATION

Governing equations

Unsteady two-dimensional MHD free convective laminar flow of incompressible viscous fluid past a vertical plate in the presence of transverse magnetic field is explored. Let Ω be a bounded/ domain and t_f be a fixed final time. The flow is taken parallel to the vertical plate in $x-$axis in upward direction, and the $y-$axis is taken normal to it.

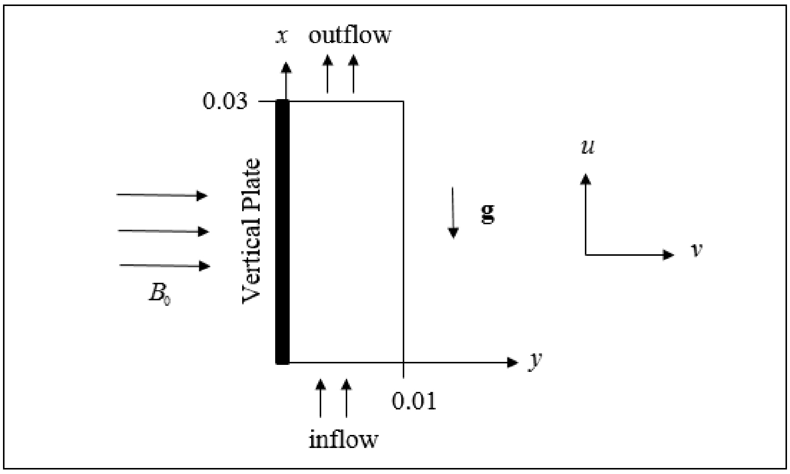

Figure 1. Flow configuration with $\Omega = [0, 0.03] \times [0, 0.01]$

The governing equations for this problem are:
ContinuityEquation:

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

Momentum Equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho g_x \beta (T - T_\infty)\tilde{i} - \sigma B_0^2 \mathbf{u} \tag{2}$$

Energy Equation:

$$\frac{\partial T}{\partial t} + \rho C_p (\mathbf{u} \cdot \nabla T) = c \nabla^2 T \tag{3}$$

where $\mathbf{u} = (u, v)$ is the fluid velocity, ρ is the fluid density, ∇p represents pressure difference due to the 'pumping' action of the flow, μ is the dynamic viscosity, σ is the electrical conductivity of the fluid, B_0 is the magnetic field strength, g_x is the gravity acceleration applied to the flow in $x-$direction, C_p is the heat capacitance of the fluid, c is the thermal conductivity of the fluid and T is the fluid temperature subject to initial and boundary conditions:

$$
\begin{aligned}
t = 0 : T = T_\infty,\ \ u = 0,\ \ v = 0,\ \ &\text{for all } x,\ y \\
t \geq 0 : T = T_w,\ \ u = 0,\ \ v = 0,\ \ &\text{for } y = 0,\ 0 \leq x \leq 0.03 \\
T = T_\infty,\ \ u = U_\infty,\ v = 0,\ \ &\text{for } y = 0.01,\ 0 \leq x \leq 0.03 \\
T = T_\infty,\ \ u = U_\infty,\ v = 0,\ \ &\text{for } x = 0,\ 0 \leq y \leq 0.01
\end{aligned}
$$

T_w is the temperature of the wall, T_∞ is the uniform temperature and U_∞ is the uniform velocity.

Next, (1) – (3) can be converted to non-dimensional form, using the following dimensionless parameters:

$$\mathbf{u} = U\mathbf{u}^*,\ t = \frac{L}{U}t^*,\ p = \rho U^2 p^*,\ \nabla = \frac{1}{L}\nabla^*,$$

$$T - T_\infty = (T_w - T_\infty)T^*$$

The dimensionless governing equations can be written as follow:

$$\nabla \cdot \mathbf{u} = 0 \tag{4}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u} + \frac{Gr}{\mathrm{Re}^2}T\tilde{i} - M\mathbf{u} \tag{5}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \frac{1}{\mathrm{Re}}\frac{1}{\mathrm{Pr}}\nabla^2 T \tag{6}$$

with corresponding dimensionless parameters defined as

$$\mathrm{Re} = \frac{UL}{\nu},\ Gr = \frac{L^3}{\nu^2}g_x\beta(T_w - T_\infty),$$

$$\mathrm{Pr} = \frac{\rho C_p \nu}{c},\ M = \frac{\sigma B^2 L}{\rho U}$$

where Re is the Reynold number, Gr is the Grashof number, Pr is the Prandtl number and M is the magnetic parameter with respect to dimensionless initial and boundary conditions:

$$
\begin{aligned}
t = 0 : T = 0,\ \ u = 0,\ \ v = 0,\ \ &\text{for all } x,\ y \\
t \geq 0 : T = 1,\ \ u = 0,\ \ v = 0,\ \ &\text{for } y = 0,\ 0 \leq x \leq 0.03 \\
T = 0,\ \ u = 1,\ \ v = 0,\ \ &\text{for } y = 0.01,\ 0 \leq x \leq 0.03 \\
T = 0,\ \ u = 1,\ \ v = 0,\ \ &\text{for } x = 0,\ 0 \leq y \leq 0.01
\end{aligned}
$$

Consistent splitting scheme

Consistent splitting scheme (CSS) is chosen to solve this time-dependent problem due to unconditionally stable for first and second order scheme (Guermond et al., 2006). Backward Euler method is used for the time derivative. Equations (4) – (6) can be written as following steps.

Step 1: We start by solving the velocity $\mathbf{u}^{n+1}$.

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \left(\mathbf{u}^n\cdot\nabla\right)\mathbf{u}^{n+1} = -\nabla p^n + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}^{n+1} + \frac{Gr}{\mathrm{Re}^2}T^n\tilde{i} - M\mathbf{u}^{n+1} \tag{7}$$

Step 2: After that, we introduce auxiliary pressure ψ.

$$\nabla\psi^{n+1} = \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \tag{8}$$

Step 3: Then, we solve the pressure p.

$$p^{n+1} = \psi^{n+1} + p^n - \frac{1}{\mathrm{Re}}\nabla\cdot\mathbf{u}^{n+1} \tag{9}$$

Step 4: Finally, we solve the temperature T.

$$\frac{T^{n+1}-T^{n}}{\Delta t}+\left(u^{n+1}\cdot\nabla\right)T^{n+1}=\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\nabla^{2}T^{n+1} \tag{10}$$

Weak formulation

In order to solve this problem using finite element method, variational form is constructed using Galerkin weighted residual approach for the discretization of governing equations. Taylor Hood element is considered to obtain the mesh of the domain for this problem where V is continuous quadratic polynomial function and Q is continuous linear polynomial function. Meanwhile, continuous quadratic polynomial function L is used for the heat transfer equation.

Equations (7) – (10) are multiplied by the test functions and integrated over the domain $\Omega=\left[0,0.03\right]\times\left[0,0.01\right]$. We have $\mathbf{u},p$ and T as functions whereas $\mathbf{v},q$ and s as test functions such that $\mathbf{u},\mathbf{v}\in \mathrm{V}$, $p,q\in \mathrm{Q}$ and $T,s\in \mathrm{L}$.

Step 1: First, we multiply (7) with test function $\mathbf{v}$. We apply integration by part and Gauss divergence theorem for second order derivatives $\nabla^{2}\mathbf{u}^{n+1}$.

$$\int_{\Omega}\left(\begin{matrix}\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n}}{\Delta t}\cdot\mathbf{v}+\left(\mathbf{u}^{n}\cdot\nabla\right)\mathbf{u}^{n+1}\cdot\mathbf{v}+\nabla p^{n}\cdot\mathbf{v}+\frac{1}{\mathrm{Re}}\nabla\mathbf{u}^{n+1}\cdot\nabla\mathbf{v}\\ -\frac{Gr}{\mathrm{Re}^{2}}T^{n}\tilde{i}\cdot\mathbf{v}+M\mathbf{u}^{n+1}\cdot\mathbf{v}\end{matrix}\right)=0 \tag{11}$$

Step 2: After that,we multiply the auxiliary pressure by a test function ∇q.

$$\int_{\Omega}\nabla\psi^{n+1}\nabla q=\int_{\Omega}\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n}}{\Delta t}\nabla q \tag{12}$$

Step 3: Then, we multiply (9) with test function q.

$$\int_{\Omega}p^{n+1}q=\int_{\Omega}\psi^{n+1}q+p^{n}q-\frac{1}{\mathrm{Re}}\left(\nabla\cdot\mathbf{u}^{n+1}\right)q \tag{13}$$

Step 4: Lastly, we multiply (10) with test function s. We apply integration by part and Gauss divergence theorem for second order derivatives $\nabla^{2}T^{n+1}$.

$$\int_{\Omega}\left(\frac{T^{n+1}-T^{n}}{\Delta t}s+\left(\mathbf{u}^{n+1}\cdot\nabla\right)T^{n+1}s-\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\nabla T^{n+1}\nabla s\right)=0 \tag{14}$$

RESULTS AND DISCUSSION

In this section, we present the investigation of free convective flow with magnetic field effects past a vertical plate. Equations (11) – (14) are solved numerically using FEniCS (Alnaes et al., 2015) and the post-processing of solutions obtained is interpreted using Paraview. The dimensionless parameter $\mathrm{Pr}=0.71, \mathrm{Re}=10^{2}$, $Gr=10^{6}$ and Richardson number for free convection $\mathrm{Ri}=Gr/\mathrm{Re}^{2}=100$ is fixed throughout the computation. We demonstrate the velocity and the temperature profiles with four different values of magnetic parameter such as $M=0,100,200$ and 500. The time step $\Delta t=0.002$ is used with 500 steps and all the numerical results are completed at $t_{f}=1$.

The grid independence test has been conducted and the results are presented in Table 1. It is shown that grid size of 64×64 gives insignificant difference with grid size of 32×32. Hence, we use this grid size to execute all results.

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Table 1. Grid independent test ($M = 0,\ t_f = 1$)

Mesh size	$\mathbf{u}_{max}$	T_{max}
8×8	1.33308708281	0.999999915505
16×16	1.33329246271	0.999999037437
32×32	1.33356932394	0.999997792891
64×64	1.33354523905	0.999993768883
128×128	1.33346003878	0.999997689405
256×256	1.33308551111	0.999988630221

Velocity profiles

The results of magnetic parameter effects on velocity profile at different time steps is presented on Figure 2. Tables 2 and 3 show the value of $\mathbf{u}(0.01, 0.001)$ at $t = 0.05, 0.1, 0.3$ and 1 while $\mathbf{u}_{max}$ at final time $t_f = 1$. It is found that the velocity profile decreases as magnetic parameter increases.

Table 2. Value of $\mathbf{u}$ at different t at $\mathbf{u}(0.01, 0.001)$

M	$t = 0.05$	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 1$
0	0.2141	0.2703	0.3263	0.3515	0.3721
100	0.2114	0.2694	0.3280	0.3556	0.3806
200	0.2090	0.2687	0.3293	0.3592	0.3887
500	0.2024	0.2667	0.3322	0.3677	0.4111

Table 3. Value of $\mathbf{u}_{max}$ at final time $t_f = 1$

M	$\mathbf{u}_{max}$
0	1.33354523905
100	1.32597268872
200	1.31884720597
500	1.29827490852

Temperature profiles

The magnetic parameter effects on temperature profile at different time steps is observed. Tables 4 and 5 show the values of $T(0.01, 0.001)$ at $t = 0.004,\ 0.01,\ 0.03$ and 1 and T_{max} at final time $t_f = 1$. It is found that no significance difference at all on temperature profile regardless of magnetic parameter values.

Table 4. Value of T at different t at $T(0.01, 0.001)$

M	$t = 0.004$	$t = 0.01$	$t = 0.03$	$t = 1$
0	0.842014	0.897386	0.891145	0.891134
100	0.842014	0.897386	0.891145	0.891134
200	0.842014	0.897386	0.891145	0.891134
500	0.842014	0.897386	0.891145	0.891134

Table 5. Value of T_{max} at final time $t_f = 1$

M	T_{max}
0	0.999993768883
100	0.999993768883
200	0.999993768883
500	0.999993768883

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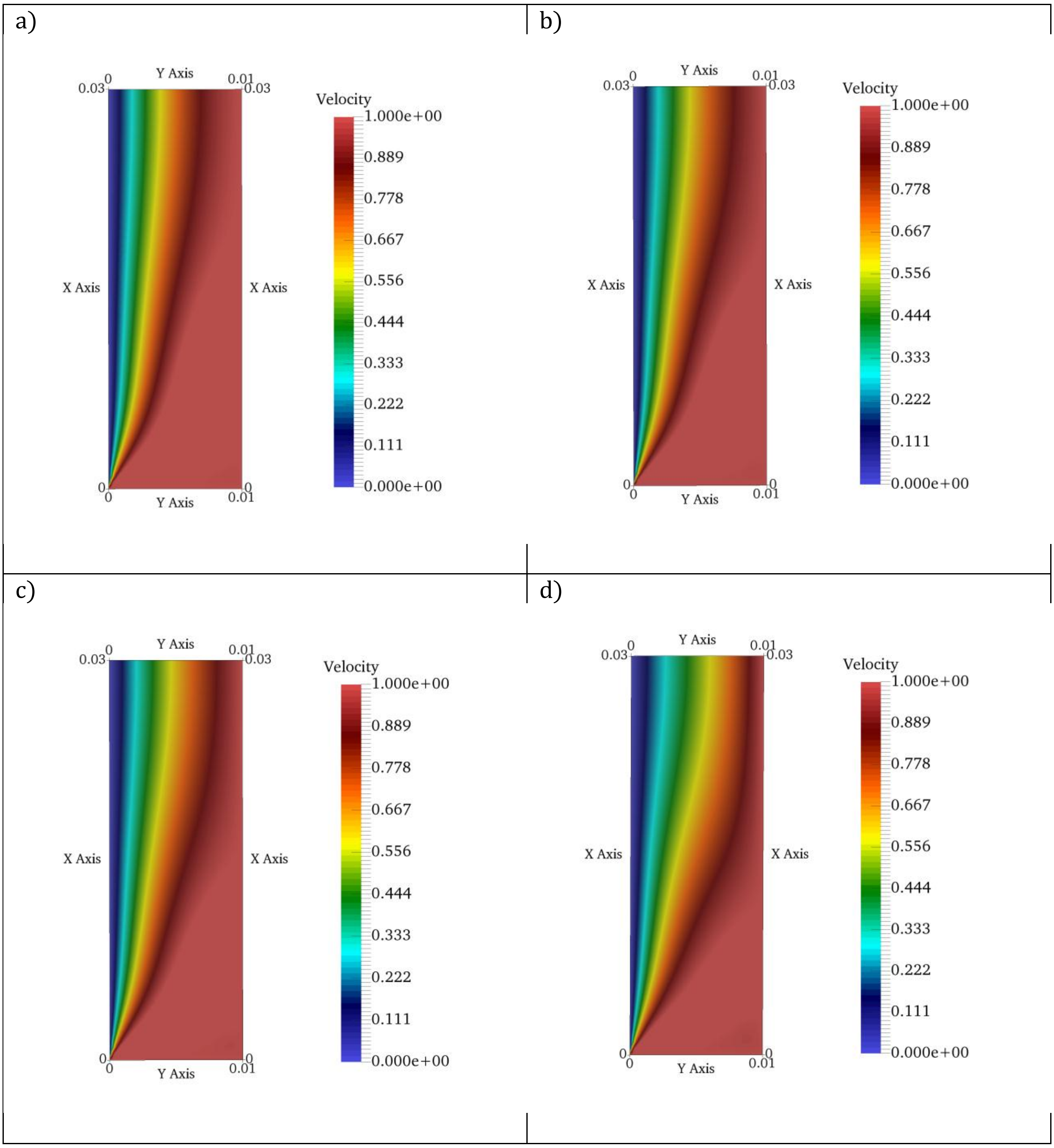

Figure 2. Velocity profiles when a) $M = 0$, b) $M = 100$ c) $M = 200$ and d) $M = 500$ at $t = 0.1$

CONCLUSION

In this paper, time dependent flow with transverse magnetic field effects past a vertical plate was considered. The dimensionless governing equations were solved using finite element method with consisting splitting scheme in FEniCS. It was found that increasing magnetic parameter where $M = 0, 100, 200$ and 500 lead to decreasing the velocity profile. Meanwhile, no significance difference for temperature profile under the presence of magnetic effect.

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